A D-dimensional Heckmann-like solution of Jordan-Brans-Dicke theory

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Abstract

In this short letter we present a some rigorous vacuum solutions of the D-dimensional Jordan-Brans-Dicke field equations. In contrast with the well known Brans-Dicke solutions [3], to the search of static and spherically symmetric space-time we choose the widespread Hilbert coordinates.

Key Words: *Jordan-Brans-Dicke theory, exact solution*

The continuing focus on extra-dimensional models in fundamental physics provides motivation for studying scalar fields in higher dimensions.The purpose of this short letter is to demonstrate that one can find the solutions in widespread Hilbert coordinates by exploiting the change of variables technique. First time, this technique was applied to a spherically symmetric case for finding a new solution to the Jordan-Brans-Dicke scalar field equations [1]. Techniques for obtaining the similar static solutions are known by know [2, 4]. In D = d + 1 dimensions, the field equations given by (we take units G = c = 1):

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{\omega}{\phi^2} \left(\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \right) + \frac{1}{\phi} \left(\phi_{\mu\nu} - g_{\mu\nu} \phi \right), \tag{1}$$

$$\nabla^{\alpha} \nabla_{\alpha} \phi = 0. \tag{2}$$

∇_α is the covariant derivative associated with the metric g, and $\nabla^\alpha \nabla_\alpha$ is the D dimensional Laplace operator of this metric, $R_{\mu\mu}$ and R are the Ricci tensor and Ricci scalar

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for an arbitrary metric g. As we have already mentioned we consider standard static and spherically symmetric space- time in Hilbert coordinates for d + 1 dimensions:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega_d^2, \tag{3}$$

where λ, ν are unknown functions of the radial coordinate r, and $d\Omega_d^2$ is the solid angle element in d - 1 dimensions. In order to simplify the problem of solving the field equations we will replace variable r by r(ν) then the field equations take a form:

$$1 + 2(d-1)\frac{r'(\nu)}{r(\nu)} - \lambda'(\nu) - \frac{2r''(\nu)}{r'(\nu)} + \frac{2\phi'(\nu)}{\phi(\nu)} = 0, \tag{4}$$

$$-1 + \lambda'(\nu)\left(1 + 2(d-1)\frac{r'(\nu)}{r(\nu)} + \frac{2\phi'(\nu)}{\phi(\nu)}\right) + \left(2 + \frac{4\phi'(\nu)}{\phi(\nu)}\right)\frac{r''(\nu)}{r'(\nu)} - 4\left(\frac{\omega\phi'(\nu)^2}{\phi(\nu)^2} + \frac{\phi''(\nu)}{\phi(\nu)}\right) = 0, \tag{5}$$

$$-1 + 2(d-2)\left(e^{\lambda(\nu)} - 1\right)\frac{r'(\nu)}{r(\nu)} + \lambda'(\nu) - \frac{2\phi'(\nu)}{\phi(\nu)} = 0, \tag{6}$$

and the equation of motion for the scalar field

$$1 + 2(d-1)\frac{r'(\nu)}{r(\nu)} - \lambda'(\nu) - \frac{2r''(\nu)}{r'(\nu)} + \frac{2\phi''(\nu)}{\phi'(\nu)} = 0, \tag{7}$$

where ν is a new variable and the primes denote derivatives with respect to ν. Making use of equation (6) in (7) they simplify to

$$\frac{\phi''(\nu)}{\phi'(\nu)} - \frac{\phi'(\nu)}{\phi(\nu)} = 0, \tag{8}$$

thus we obtain

$$\phi(\nu) = \alpha e^{\beta}, \tag{9}$$

where α and β is a arbitrary constants. Using the asymptotic condition in infinity we have $\phi(\nu)$ =1. In the case β = 0 one can find solution of equations (4) - (7)

$$r(\nu) = \frac{const}{e^{\nu} - 1}, \; e^{\lambda(\nu)} = e^{-\nu}, \; \phi(\nu) = 1, \tag{10}$$

that identical to the Schwarzschild solution of the Einstein theory. For the more general case $\beta \neq 0$ making use equations (4) , (6) and (9) we eliminate $\lambda'(\nu)$ and obtain for r (ν)

$$r(\nu) = \xi e^{-\frac{2\beta(\omega\beta-1)}{\Phi^2+\Psi^2}\left(\Phi\zeta+(1+2\beta)\nu+2\arctan h\left[\frac{\Psi\tan\left[\frac{1}{2}\left(\zeta+\frac{\nu}{\sqrt{d-1}}\right)\right]}{\Phi}\right]\right)} \tag{11}$$

$$\left(\Phi^2 - \Psi^2 + \left(\Phi^2 + \Psi^2\right)\right)\cos\left[\left(\zeta + \frac{\nu}{\sqrt{d-1}}\right)\Psi\right]^{\frac{2\beta(\omega\beta-1)}{\Phi^2+\Psi^2}}, \tag{12}$$

where ζ and ξ is a arbitrary constants, and

$$\Phi = (1+2\beta)\sqrt{d-1}, \tag{13}$$

$$\Psi = \sqrt{1+4\beta\left(2\beta\omega+\beta-1\right)-d\left(1+4\beta^2\left(1+\omega\right)\right)}, \tag{14}$$

After same algebra one can find from (5) - (6) for $\lambda\left(\nu\right)$:

$$\lambda\left(\nu\right) = \ln\left[-\frac{\Psi^2 \sec\left(\frac{\left(\sqrt{d-1}\zeta+\nu\right)\Psi}{2\sqrt{d-1}}\right)^2}{4\left(d-2\right)\left(\beta\omega-1\right)}\right], \tag{15}$$

Notice that for some special circumstances one can express these solutions in more natural view concerning a variable r. First let us consider that we can express the field equations for the case $\phi = \alpha e^{\frac{\nu}{\omega}}$, then

$$\begin{aligned} e^{\nu} &= \chi\, r^{-\frac{d\omega}{2+\omega}}\left(\left(2-d\right)r^d\sigma\omega + \left(2+\omega\right)r^2\right)^{\frac{\omega}{2+\omega}}, \\ e^{\lambda} &= \frac{1}{1-\frac{2+\omega}{(d-2)\sigma\omega}r^{2-d}}, \\ \phi &= \alpha e^{\frac{\nu}{\omega}}. \end{aligned} \tag{16}$$

or

$$\begin{aligned} e^{\nu} &= \left(\chi\ + \left(\frac{r}{\sigma}\right)^{2-d}\right)^{\frac{\omega}{2+\omega}}, \\ e^{\lambda} &= \frac{\chi}{\chi-\left(\frac{r}{\sigma}\right)^{2-d}}, \\ \phi &= \alpha e^{\frac{\nu}{\omega}}. \end{aligned} \tag{17}$$

where χ and σ is a arbitrary constants. On the other hand one can write the $\phi = \alpha e^{-\frac{\nu}{2}}$, then

$$\begin{aligned} e^{\nu} &= const\left(\left(\frac{r}{\sigma}\right)^{2-d} + \sqrt{1+\left(\frac{r}{\sigma}\right)^{2(2-d)}}\right)^{2\sqrt{\frac{d-1}{(2+\omega)(d-2)}}}, \\ e^{\lambda} &= \frac{1}{1+\left(\frac{r}{\sigma}\right)^{2(2-d)}}, \\ \phi &= \alpha e^{-\frac{\nu}{2}}. \end{aligned} \tag{18}$$

Another choice of function $\nu = a\left(b + e^{-\lambda/2}\right)$ would lead to a different solution

$$e^{\nu} = e^{a\left(1+b+\Pr oductLog\left(\eta\ r^{2-d}\right)\right)},$$

$$e^{\lambda} = \frac{1}{\left(1+\Pr oductLog\left(\eta\ r^{2-d}\right)\right)^{2}}, \tag{19}$$

$$\phi = \gamma e^{-\frac{1}{2}(a-2)\Pr oductLog\left(\eta\ r^{2-d}\right)}.$$

where η and γ is a arbitrary constants and

$$a = \frac{2}{(2+\omega)}\left(1+\omega+\sqrt{\frac{\omega - d\,(1+\omega)}{d-2}}\right). \tag{20}$$

Conclusion

In this paper D - dimensional spherically symmetric static solutions of the Jordan-Brans-Dicke field equations are investigated. This analysis shows that the solutions can be explicitly written down in Hilbert coordinates, contrary to what it is usually claimed in the literature. A number of higher dimensional generalizations of the Heckmann solution have been found.

References

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